The smallest eigenvalue of Hankel matrices

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Abstract

Let $\mathcal{H}_N = (s_{n+m}), n, m \le N$ denote the Hankel matrix of moments of a positive measure with moments of any order. We study the large N behaviour of the smallest eigenvalue λ_N of $\mathcal{H}_N$. It is proved that λ_N has exponential decay to zero for any measure with compact support. For general determinate moment problems the decay to 0 of λ_N can be arbitrarily slow or arbitrarily fast. In the indeterminate case, where λ_N is known to be bounded below, we prove that the limit of the n'th smallest eigenvalue of $\mathcal{H}_N$ for $N \to \infty$ tends rapidly to infinity with n. The special case of the Stieltjes-Wigert polynomials is discussed.

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1 Introduction

Let (s_n) be the moment sequence of a positive measure μ on $\mathbb{R}$ with infinite support,

$$s_n = \int x^n \, d\mu(x), \quad n \ge 0. \tag{1}$$

By Hamburger's theorem this is equivalent to a real sequence (s_n) such that all the Hankel matrices

$$\mathcal{H}_N = (s_{n+m})_{n,m=0}^{N}, \quad N = 0, 1, \ldots \tag{2}$$

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are positive definite. The smallest eigenvalue of $\mathcal{H}_N$ is the positive number

$$\lambda_N = \min\{\langle \mathcal{H}_N a, a\rangle \mid a \in \mathbb{C}^{N+1}, ||a|| = 1\}, \tag{3}$$

and clearly $\lambda_0 \geq \lambda_1 \geq \ldots$. The large N behaviour of λ_N has been studied in the papers [4, 8, 9, 18, 22, 24]. See also results in [2, 14] about the behaviour of the condition number $\kappa(\mathcal{H}_N) = \Lambda_N/\lambda_N$, where Λ_N denotes the largest eigenvalue of $\mathcal{H}_N$.

Widom and Wilf [22] found the asymptotic behaviour

$$\lambda_N \sim AN^{1/2}B^N, \tag{4}$$

for certain constants $A > 0, 0 < B < 1$ in the case of a measure μ of compact support in the Szegő class, generalizing results by Szegő [18]. In the same paper Szegő also obtained results about the Hermite and Laguerre case, namely

$$\lambda_N \sim AN^{1/4}B^{N^{1/2}}, \tag{5}$$

again with certain A, B as above. In all of this paper $a_N \sim b_N$ means that $a_N/b_N \to 1$ as $N \to \infty$.

Chen and Lawrence [8] found the asymptotic behaviour of λ_N for the case of μ having the density e^{-t^β} with respect to Lebesgue measure on the interval $[0, \infty[$. The result requires $\beta > 1/2$, and we refer to [8] for the quite involved expression. For $\beta = \frac{1}{2}$ the asymptotic behaviour is only stated as a conjecture:

$$\lambda_N \sim A\frac{\sqrt{\log N}}{N^{2/\pi}}$$

for a certain constant $A > 0$.

Chen and Lubinsky [9] found the asymptotic behaviour of λ_N, when μ is a generalized (symmetric) exponential weight including $e^{-|x|^\alpha}$ with $\alpha > 1$.

We recall that the density e^{-t^β} on the half-line is determinate for $\beta \geq \frac{1}{2}$, i.e. there are no other measures having the moments

$$s_n = \int_0^\infty t^n e^{-t^\beta}\, dt = \Gamma\left(\frac{n+1}{\beta}\right)/\beta. \tag{6}$$

However, for $0 < \beta < \frac{1}{2}$ the density is Stieltjes indeterminate: There are infinitely many measures on the half-line with the moments (6). The symmetric density $e^{-|x|^\alpha}$ is determinate if and only if $\alpha \geq 1$. For general information about the moment problem see [1, 15, 16].

Berg, Chen and Ismail proved in [4] the general result that the moment sequence (1) (or the measure μ) is determinate if and only if $\lambda_N \to 0$ for $N \to \infty$ and found the positive lower bound $\lambda_N \geq 1/\rho_0$ in the indeterminate case, where ρ_0 is given in (15) below.

The purpose of the present paper is to prove some general results about the behaviour of λ_N.

In section 2 we prove that λ_N tends to zero exponentially for any measure μ of compact support. Theorem 2.3 is a slightly sharpened version, where only the boundedness of the coefficients (b_n) from the three term recurrence relation (7) is assumed. We also show that λ_N may tend to zero arbitrarily fast.

Section 3 is devoted to showing that there exist determinate measures for which λ_N tends to zero arbitrarily slowly, cf. Theorem 3.6.

In Section 4 we consider the indeterminate case, where λ_N is bounded below by a positive constant. We prove that the n'th smallest eigenvalue $\lambda_{N,n}$ of (2) $(n \le N)$ has a lower bound $\lambda_{\infty,n} = \lim_{N\to\infty} \lambda_{N,n}$,which tends rapidly to infinity with n, cf. Theorem 4.4. To describe our results in detail we need some more notation.

We let (P_n) denote the sequence of orthonormal polynomials with respect to μ, uniquely determined by the requirements that P_n is a polynomial of degree n with positive leading coefficient and the orthonormality condition $\int P_n P_m \, d\mu = \delta_{nm}$.

The orthonormal polynomials satisfy the following three-term recurrence relation

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x), \tag{7}$$

where $b_n > 0$ and $a_n \in \mathbb{R}$.

We need the coefficients of the orthonormal polynomials (P_n) with respect to μ:

$$P_n(x) = \sum_{k=0}^{n} b_{k,n} x^k, \tag{8}$$

and consider the infinite upper triangular matrix

$$\mathcal{B} = (b_{k,n}), \qquad b_{k,n} = 0,\ k > n. \tag{9}$$

Let $\mathcal{B}_N$ denote the $(N+1)\times(N+1)$-matrix obtained from $\mathcal{B}$ by assuming $k, n \le N$ and let $\mathcal{A}^{(N)} = \mathcal{B}_N \mathcal{B}_N^*$. Defining the kernel polynomial

$$K_N(z,w) = \sum_{n=0}^{N} P_n(z)P_n(w) = \sum_{j,k=0}^{N} \left(\sum_{n=\max(j,k)}^{N} b_{j,n} b_{k,n} \right) z^j w^k, \tag{10}$$

we see that $\mathcal{A}^{(N)} = (a_{j,k}^{(N)})$ is the $(N+1)\times(N+1)$-matrix of coefficients to $z^j w^k$ in $K_N(z,w)$. The following result going back to A.C. Aitken, cf. Collar [12], has been rediscovered several times, see [3, 17].

Theorem 1.1.

$$\mathcal{A}^{(N)} = \mathcal{H}_N^{-1}.$$

For completeness we give the simple proof of Theorem 1.1:
For $0 \le k \le N$ we have by the reproducing property

$$\int x^k K_N(x,y)\,d\mu(x) = y^k.$$

On the other hand we have

$$\int x^k K_N(x,y)\,d\mu(x) = \sum_{j=0}^{N} (\sum_{\ell=0}^{N} s_{k+\ell} a_{\ell,j}^{(N)}) y^j,$$

and therefore

$$\sum_{\ell=0}^{N} s_{k+\ell} a_{\ell,j}^{(N)} = \delta_{k,j}. \quad \square$$

The following Lemma is also very simple. The identity matrix is denoted $I = (\delta_{j,k})$.

Lemma 1.2. *As infinite matrices we have*

$$\mathcal{B}(\mathcal{B}^*\mathcal{H}) = (\mathcal{B}^*\mathcal{H})\mathcal{B} = \mathcal{B}^*(\mathcal{H}\mathcal{B}) = I,$$

and $\mathcal{B}^*\mathcal{H}$ is an upper triangular matrix.

Proof. The matrix products $\mathcal{B}^*\mathcal{H}$ and $\mathcal{H}\mathcal{B}$ are well-defined because $\mathcal{B}$ is upper triangular, and we get

$$(\mathcal{B}^*\mathcal{H})_{j,k} = \sum_{n=0}^{j} b_{n,j} s_{n+k} = \int P_j(x) x^k \, d\mu(x),$$

which is clearly 0 for $j > k$, so $\mathcal{B}^*\mathcal{H}$ is also upper triangular. Therefore, $\mathcal{B}(\mathcal{B}^*\mathcal{H})$ is well-defined and upper triangular. For $l \le k$ we finally get

$$(\mathcal{B}(\mathcal{B}^*\mathcal{H}))_{l,k} = \sum_{j=0}^{k} b_{l,j} \sum_{n=0}^{j} b_{n,j} s_{n+k} = \sum_{n=0}^{k} \left(\sum_{j=0}^{k} b_{l,j} b_{n,j} \right) s_{n+k} = \delta_{l,k}$$

by Theorem 1.1 with $N = k$.

The relation $(\mathcal{B}^*\mathcal{H})\mathcal{B} = \mathcal{B}^*(\mathcal{H}\mathcal{B}) = I$ is an easy consequence of the orthogonality of (P_n) with respect to μ. $\square$

We also consider the infinite matrix

$$\mathcal{K} = (\kappa_{j,k}), \quad \kappa_{j,k} = \frac{1}{2\pi} \int_0^{2\pi} P_j(e^{it}) P_k(e^{-it}) \, dt. \tag{11}$$

It is a classical fact that the indeterminate case occurs if and only if

$$\sum_{n=0}^{\infty} |P_n(z)|^2 < \infty \tag{12}$$

for all $z \in \mathbb{C}$. It suffices that (12) holds for just one point $z_0 \in \mathbb{C} \setminus \mathbb{R}$, and in this case the convergence of (12) is uniform on compact subsets of the complex plane.

In the indeterminate case we can let $N \to \infty$ in (10) leading to the entire function of two complex variables

$$K(z,w) = \sum_{n=0}^{\infty} P_n(z) P_n(w) = \sum_{j,k=0}^{\infty} a_{j,k} z^j w^k, \tag{13}$$

and we collect the coefficients of the power series as the symmetric matrix

$$\mathcal{A} = (a_{j,k}). \tag{14}$$

In Proposition 4.2 we prove that the matrices $\mathcal{A}, \mathcal{B}, \mathcal{K}$ are of trace class in the indeterminate case and

$$\mathrm{tr}\,(\mathcal{A}) = \mathrm{tr}\,(\mathcal{K}) = \rho_0,$$

where ρ_0 is given by

$$\rho_0 = \frac{1}{2\pi} \int_0^{2\pi} K(e^{it}, e^{-it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{\infty} \left| P_k\left(\mathrm{e}^{it}\right) \right|^2 dt < \infty. \tag{15}$$

In the indeterminate case the infinite Hankel matrix $\mathcal{H} = (s_{n+m})$ does not correspond to an operator on ℓ^2 defined on $\mathrm{span}\{\delta_n | n \geq 0\}$. In fact, by Carleman's theorem we necessarily have $\sum_{n=0}^{\infty} s_{2n}^{-1/(2n)} < \infty$, hence $s_{2n} \geq 1$ for n sufficiently large, and therefore

$$\sum_{m=0}^{\infty} s_{n+m}^2 = \infty \text{ for all } n.$$

It is likely that Theorem 1.1 extends to the indeterminate case in the sense that $\mathcal{AH} = \mathcal{HA} = I$, where the infinite series $\sum_l a_{k,l} s_{l+j}$ defining $\mathcal{AH}$ and $\mathcal{HA}$ are absolutely convergent. We have not been able to prove this general statement, but it holds for the Stieltjes-Wigert case which is treated in Section 5.

The Stieltjes-Wigert polynomials $P_n(x;q)$ are defined in (50). They are orthogonal with respect to a log-normal distribution, known to be indeterminate, and the corresponding moment sequence is $s_n = q^{-(n+1)^2/2}$. It is known that the modified moment sequence $(\tilde{s}_n)$ given by $\tilde{s}_n = s_n$ for $n \geq 1$ and

$$\tilde{s}_0 = s_0 - (\sum_{n=0}^{\infty} P_n(0;q)^2)^{-1}$$

is determinate, and the corresponding measure $\tilde{\mu}$ is discrete given by

$$\tilde{\mu} = \sum_{x \in X} c_x \delta_x, \tag{16}$$

where X is the zero set of the reproducing kernel $K(0, z)$ defined in (13) and

$$c_x = \left(\sum_{k=0}^{\infty} P_k(x;q)^2 \right)^{-1}, \quad x \in X. \tag{17}$$

The Hankel matrices $\mathcal{H} = (s_{j+k})$ and $\tilde{\mathcal{H}} = (\tilde{s}_{j+k})$ agree except for the upper left corner. In Theorem 5.2 we prove that the smallest eigenvalue $\tilde{\lambda}_N$ of the Hankel matrix $\tilde{\mathcal{H}}_N$ tends to zero exponentially (while λ_N is bounded below). We do it by determining the corresponding orthonormal polynomials $\tilde{P}_n(x;q)$, see Theorem 5.3.

2 Fast decay

We start by proving a lemma which is essentially contained in [4, §2].

Lemma 2.1. *For each $z_0 \in \mathbb{C}$ with $|z_0| < 1$ we have*

$$\lambda_N \leq \left((1 - |z_0|^2) \sum_{n=0}^{N} |P_n(z_0)|^2 \right)^{-1}. \tag{18}$$

Proof. For any $a \in \mathbb{C}^{N+1}, a \neq 0$ we have by (3)

$$\lambda_N \leq \frac{\langle \mathcal{H}_N a, a \rangle}{||a||^2}.$$

This means that for any non-zero polynomial

$$p(x) = \sum_{k=0}^{N} a_k x^k = \sum_{n=0}^{N} c_n P_n(x)$$

we have

$$\lambda_N \leq \frac{\int |p|^2 \, d\mu}{\frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^2 \, dt}. \tag{19}$$

Moreover, by Cauchy's integral formula

$$p(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{p(e^{it}) e^{it}}{e^{it} - z_0} \, dt,$$

hence by Cauchy-Schwarz's inequality

$$|p(z_0)|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |p(e^{it})|^2\, dt\, \frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{|e^{it}-z_0|^2}, \tag{20}$$

and the last integral equals $(1-|z_0|^2)^{-1}$ by a well-known property of the Poisson kernel. Combining (19) and (20) for the polynomial

$$p(x) = \sum_{n=0}^{N} \overline{P_n(z_0)} P_n(x)$$

leads to

$$\lambda_N \leq \frac{\sum_{n=0}^N |P_n(z_0)|^2}{(1-|z_0|^2)|p(z_0)|^2} = \left((1-|z_0|^2)\sum_{n=0}^{N} |P_n(z_0)|^2\right)^{-1}.$$

□

Remark 2.2. It follows immediately from Lemma 2.1 that if $\lambda_N \geq c > 0$ for all N, then

$$\sum_{n=0}^{\infty} |P_n(z_0)|^2 < \infty$$

for all z_0 with $|z_0| < 1$, hence (s_n) is indeterminate.

The following theorem proves that λ_N tends to zero exponentially in the sense that there is an estimate of the form

$$\lambda_N \leq AB^N, \quad A > 0, 0 < B < 1, \tag{21}$$

whenever the measure μ in (1) has compact support.

Theorem 2.3. *Assume that the sequence (b_n) from (7) is bounded with $b := \limsup b_n$. Then*

$$\limsup \lambda_N^{1/N} \leq \frac{2b^2}{1+2b^2}.$$

Remark 2.4. Notice that the condition $\limsup b_n < \infty$ implies that $\sum 1/b_n = \infty$, so by Carleman's theorem the moment problem is determinate, cf. [1, p.24]. We also recall the fact that μ has compact support if and only if $(a_n), (b_n)$ from (7) are bounded sequences.

Proof. Taking $z_0 = \alpha i$, where $0 < \alpha < 1$, we obtain from Lemma 2.1

$$\lambda_N \leq \left((1-\alpha^2)\sum_{n=0}^{N} |P_n(\alpha i)|^2\right)^{-1} \leq \left((1-\alpha^2)[|P_{N-1}(\alpha i)|^2 + |P_N(\alpha i)|^2]\right)^{-1}.$$

Since the distance from the point αi to the support of the orthogonality measure is at least α, we obtain by [20, Remark 2, p. 148]

$$\limsup \lambda_N^{1/N} \le \frac{1}{1+\frac{\alpha^2}{2b^2}} = \frac{2b^2}{\alpha^2+2b^2}.$$

As α is an arbitrary number less than 1 we get

$$\limsup \lambda_N^{1/N} \le \frac{2b^2}{1+2b^2}.$$

□

Theorem 2.5. *For any decreasing sequence (τ_n) of positive numbers with $\tau_0 = 1$ and $\lim \tau_n = 0$, there exist determinate probability measures μ for which $\lambda_N \le \tau_N$ for all N.*

Proof. We will construct symmetric probability measures μ with the desired property. Let

$$xP_n(x) = b_n P_{n+1}(x) + b_{n-1}P_{n-1}(x) \tag{22}$$

be the three-term recurrence relation for the orthonormal polynomials associated with a symmetric μ. We shall choose $b_n > 0, n \ge 0$ such that $\lambda_N \le \tau_N$ for all $N \ge 0$. We always have $\lambda_0 = \tau_0 = 1$ because μ is a probability measure. Since $s_1 = 0$ we know that $\lambda_1 = \min(1, s_2), s_2 = b_0^2$, so we can choose $0 < b_0 \le 1$ such that $\lambda_1 = \tau_1$.

By Lemma 2.1 with $z_0 = 0$ we get

$$\lambda_N \le \left(\sum_{n=0}^{N} |P_n(0)|^2\right)^{-1},$$

and in particular

$$\lambda_{2N+1} \le \lambda_{2N} \le \frac{1}{P_{2N}^2(0)}. \tag{23}$$

By (22) we have

$$P_{2n}(0) = (-1)^n \frac{b_0 b_2 \dots b_{2n-2}}{b_1 b_3 \dots b_{2n-1}}, \quad n \ge 1,$$

and defining

$$r_k = \frac{b_{2k-1}}{b_{2k-2}}, \quad k \ge 1$$

we get

$$\lambda_{2N+1} \le \lambda_{2N} \le r_1^2 r_2^2 \dots r_N^2, \quad N \ge 1,$$

and we will choose $r_k, k \ge 1$, such that

$$r_1^2 r_2^2 \dots r_N^2 \le \tau_{2N+1}, \quad N \ge 1.$$

First choose $0 < r_1 \le \sqrt{\tau_3}$, and when $r_1, \dots, r_{N-1}$ have been chosen, we choose

$$0 < r_N \le \min\left(1, \frac{\sqrt{\tau_{2N+1}}}{r_1 \dots r_{N-1}}\right).$$

It is clear that the sequence (r_k) can be chosen such that $r_k \to 0$. We next define $b_1 = r_1 b_0$ and we finally have an infinity of choices of $b_{2k-1}, b_{2k-2} > 0$ to satisfy $r_k = b_{2k-1}/b_{2k-2}, k \ge 2$.

If (r_k) converges to zero, the decay of λ_n is faster than exponential. Clearly the corresponding moment problem is determinate since

$$|P_{2n}(0)| = (r_1 r_2 \dots r_n)^{-1} \ge 1.$$

In particular, the unique measure μ solving the moment problem carries no mass at 0. □

After having chosen the numbers r_k we have several possibilities for selecting the coefficients b_n. We will discuss three such choices.

Example 1. For $k \ge 2$ let $b_{2k-2} = 1$ and $b_{2k-1} = r_k$ and assume that $r_k \to 0$. Then the corresponding Jacobi matrix J is bounded and it acts on ℓ^2 by

$$(Jx)_n = b_n x_{n+1} + b_{n-1} x_{n-1}, \quad x = (x_n).$$

Let us compute the square of J. We have

$$(J^2 x)_n = b_n b_{n+1} x_{n+2} + (b_{n-1}^2 + b_n^2) x_n + b_{n-2} b_{n-1} x_{n-2}.$$

By the choice of (b_n) we get $b_n b_{n+1} \to 0$ and $b_{n-1}^2 + b_n^2 \to 1$. Therefore the operator J^2 is of the form $J^2 = I + K$, where K is a compact operator. Hence its spectrum consists of a sequence of positive numbers converging to 1. Thus the spectrum of J is of the form $\sigma(J) = \{\pm t_n\}$, where t_n is a sequence of positive numbers converging to 1, so the measure μ is discrete with bounded support.

Example 2. Let $b_{2k-2} = r_k^{-1}$ and $b_{2k-1} = 1$ and assume $r_k \to 0$. Then the corresponding Jacobi matrix J is unbounded. By the recurrence relation we have

$$x^2 P_{2n}(x) = b_{2n} b_{2n+1} P_{2n+2}(x) + (b_{2n-1}^2 + b_{2n}^2) P_{2n}(x) + b_{2n-2} b_{2n-1} P_{2n-2}(x). \quad (24)$$

Then $Q_n(y) = P_{2n}(\sqrt{y})$ is a polynomial of degree n satisfying

$$y Q_n(y) = r_{n+1}^{-1} Q_{n+1}(y) + (1 + r_{n+1}^{-2}) Q_n(y) + r_n^{-1} Q_{n-1}(y).$$

Letting $B_n = r_n^{-1}$ and $A_n = (1 + r_{n+1}^{-2})$ we get

$$\frac{B_n^2}{A_{n-1} A_n} = \frac{r_{n+1}^2}{(1 + r_n^2)(1 + r_{n+1}^2)} \xrightarrow{n} 0,$$

so by Chihara's Theorem (see [10, Th. 8] and [21, Theorem 2.6]) we see that the orthogonality measure ν for $Q_n(y)$ is discrete. However, ν is the image measure of the symmetric measure μ under the mapping $x \to x^2$, so also μ is discrete with unbounded support.

Example 3. Let $b_{2k-2} = r_k^{-1/2}$ and $b_{2k-1} = r_k^{1/2}$. With $Q_n(y) = P_{2n}(\sqrt{y})$ as in Example 2 we get from (24)

$$yQ_n(y) = Q_{n+1}(y) + a_n Q_n(y) + Q_{n-1}(y)$$

where $a_n = r_n + 1/r_{n+1}$. If $r_k \to 0$ we see again that μ is discrete with unbounded support.

3 Slow decay

The goal of this section is to prove that there exist moment sequences (s_n) such that the corresponding sequence (λ_N) from (3) tends to 0 arbitrarily slowly. This is proved in Theorem 3.6.

Consider a symmetric probability measure μ on the real line with moments of any order and infinite support. The corresponding orthonormal polynomials (P_n) satisfy a symmetric recurrence relation (22), where $b_n > 0$ for $n \geq 0$. For simplicity we assume that the second moment of μ is 1, i.e. $s_2 = b_0^2 = 1$ and hence $\lambda_0 = \lambda_1 = 1$. This can always be achieved by replacing $d\mu(x)$ by $d\mu(ax)$ for suitable $a > 0$. Note that $P_0 = 1, P_1(x) = x$ in this case.

Lemma 3.1. *Let (P_n) denote the orthonormal polynomials satisfying (22) with $b_0 = 1$. The sequence*

$$u_n = |P_n(i)|, \quad n \geq 0 \tag{25}$$

satisfies

$$u_{n+1} = \frac{1}{b_n} u_n + \frac{b_{n-1}}{b_n} u_{n-1}, \quad n \geq 1, \tag{26}$$

with $u_0 = u_1 = 1$. Moreover, for $n \geq 0$

$$|P_n(z)| \leq u_n, \quad |z| \leq 1.$$

Proof. Let $k_n = b_{n,n}$ denote the (positive) leading coefficient of P_n and let $x_1, x_2 \ldots, x_n$ denote the positive zeros of P_{2n}. Then

$$P_{2n}(x) = k_{2n}(x^2 - x_1^2)(x^2 - x_2^2) \ldots (x^2 - x_n^2),$$

hence

$$u_{2n} = (-1)^n P_{2n}(i) > 0. \tag{27}$$

Similarly, let $y_1, y_2 \ldots, y_n$ denote the positive zeros of P_{2n+1}. Then

$$P_{2n+1}(x) = k_{2n+1} x(x^2 - y_1^2)(x^2 - y_2^2) \ldots (x^2 - y_n^2),$$

hence

$$u_{2n+1} = (-1)^{n+1} i\, P_{2n+1}(i) > 0. \tag{28}$$

Combining (22), (27) and (28) gives (26).

By (22) we get for $|z| \le 1$

$$|P_{n+1}(z)| \le \frac{1}{b_n}\,|P_n(z)| + \frac{b_{n-1}}{b_n}\,|P_{n-1}(z)|.$$

Therefore, (26) can be used to show by induction that $|P_n(z)| \le u_n$. $\square$

Proposition 3.2. *Assume that the coefficients* (b_n) *from* (22) *satisfy* $b_0 = 1$ *and* $b_{n-1} + 1 \le b_n$*, for* $n \ge 1$ *and let* $u_n = |P_n(i)|$*. Then*

$$\max(u_{2n}, u_{2n+1}) \le \prod_{k=1}^{n} \max\left(\frac{1 + b_{2k-2}}{b_{2k-1}}, \frac{1 + b_{2k-1}}{b_{2k}}\right), \quad n \ge 0.$$

Proof. Since

$$\frac{1 + b_{k-1}}{b_k} \le 1,$$

we get from (26)

$$u_{k+1} \le \max(u_{k-1}, u_k), \quad k \ge 1.$$

We have clearly

$$u_k \le \max(u_{k-1}, u_k),$$

thus

$$\max(u_k, u_{k+1}) \le \max(u_{k-1}, u_k), \quad k \ge 1.$$

This implies by (26)

$$u_{n+1} \le \frac{1 + b_{n-1}}{b_n} \max(u_{n-1}, u_n) \le \frac{1 + b_{n-1}}{b_n} \max(u_{n-2}, u_{n-1}),$$

and replacing n by $n-1$ in the first inequality

$$u_n \le \frac{1 + b_{n-2}}{b_{n-1}} \max(u_{n-2}, u_{n-1}).$$

Combining the last two inequalities gives

$$\max(u_n, u_{n+1}) \le \max\left(\frac{1 + b_{n-2}}{b_{n-1}}, \frac{1 + b_{n-1}}{b_n}\right) \max(u_{n-2}, u_{n-1}), \quad n \ge 2,$$

which implies the conclusion because $u_0 = u_1 = 1$. $\square$

Lemma 3.3. *Let (b_n) and (u_n) be as in Proposition 3.2. Then the sequence of eigenvalues (λ_N) from (3) satisfies*

$$\lambda_N \geq \left(\sum_{k=0}^{N} u_k^2\right)^{-1}.$$

Proof. By [4, (1.12)] we have

$$\lambda_N \geq \left(\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{N}|P_k(e^{it})|^2\,dt\right)^{-1}.$$

The conclusion follows now by Lemma 3.1, which shows that $|P_k(e^{it})| \leq u_k$. □

Using the assumption of Proposition 3.2, we adopt the notation

$$1-\eta_k = \max\left(\frac{1+b_{2k-2}}{b_{2k-1}}, \frac{1+b_{2k-1}}{b_{2k}}\right), \quad k \geq 1. \tag{29}$$

Proposition 3.4. *Let (b_n) and (u_n) be as in Proposition 3.2. Then the sequence of eigenvalues (λ_N) from (3) satisfies*

$$\lambda_{2N+1} \geq \left(2+2\sum_{k=1}^{N}\prod_{l=1}^{k}(1-\eta_l)^2\right)^{-1}.$$

Proof. By Lemma 3.3 and the fact that $u_0 = u_1 = 1$ we have

$$\lambda_{2N+1} \geq \left(2+\sum_{k=1}^{N}(u_{2k}^2+u_{2k+1}^2)\right)^{-1}.$$

Proposition 3.2 states that

$$\max(u_{2k}, u_{2k+1}) \leq \prod_{l=1}^{k}(1-\eta_l).$$

These two inequalities give the conclusion. □

Lemma 3.5. *Let (b_n) be as in Proposition 3.2 and define ξ_n by*

$$\frac{b_{n-1}+1}{b_n} = 1-\xi_n, \quad n \geq 1.$$

Then

$$b_n = \prod_{k=1}^{n}(1-\xi_k)^{-1}\left[2+\sum_{k=1}^{n-1}\prod_{l=1}^{k}(1-\xi_l)\right], \quad n \geq 1. \tag{30}$$

Proof. We have

$$b_n = (1-\xi_n)^{-1}(1+b_{n-1}) = (1-\xi_n)^{-1}\left(1+(1-\xi_{n-1})^{-1}(1+b_{n-2})\right) = \dots,$$

and after n steps the formula ends using $b_0 + 1 = 2$. $\square$

Theorem 3.6. *Let (τ_n) be a decreasing sequence of positive numbers satisfying $\tau_n \to 0$ and $\tau_0 < 1$. Then there exists a determinate symmetric probability measure μ on $\mathbb{R}$ for which $\lambda_N \geq \frac{1}{2}\tau_N$ for all N.*

In other words, the eigenvalues λ_N can decay arbitrarily slowly.

The proof depends on the following

Lemma 3.7. *Let (e_n) be an increasing sequence of positive numbers such that $e_0 > 1$ and $\lim e_n = \infty$. There exists a strictly increasing concave sequence (d_n) such that $d_0 = 1$, $d_n \leq e_n$ for all n and $\lim d_n = \infty$.*

Proof. Define a function $f(x)$ on $[0,\infty)$ by $f(0) = e_0$ and $f(x) = e_n$ for $n-1 < x \leq n$, for $n \geq 1$. This function is left continuous. The discontinuity points in $]0,\infty[$ are denoted by e_{n_k} for a strictly increasing subsequence n_k of natural numbers. Consider the sequence A_k of points in the plane given by $A_0{=}(0,1)$ and $A_k = (n_k, e_{n_k})$ for $k \geq 1$. If we connect every two consecutive points A_k and A_{k+1} by the line segment we will obtain a graph of a strictly increasing piecewise linear function $g(x)$ such that $g(x) \leq f(x)$. Moreover $g(x)$ tends to infinity at infinity. We are going to construct the graph of a concave function $h(x)$ such that $h(x) \leq g(x)$, $h(0) = 1$ and $h(x) \to \infty$ as $x \to \infty$. Once it is done the sequence $d_n = h(n)$ satisfies the conclusion of the lemma. We will construct the graph of $h(x)$ by tracing the graph Γ of $g(x)$. The points of Γ where the slope changes will be called nodes.

We start at the point $(0,1)$ and draw a graph of the function $h(x)$. We go along the first line segment of Γ until we reach the first node. Then we inspect the slope of the next line segment of Γ. If it is smaller than the slope of the previous segment we continue along Γ until we reach the next node. Otherwise we do not change slope and continue drawing the straight line (below Γ). In this case two possibilities may occur. The line does not hit Γ. Then the graph of $h(x)$ is constructed. Otherwise the line hits Γ. Then two cases are considered. If the line hits a node of Γ, then we follow the procedure described above for the first node. If the line hits an interior point of a segment γ of Γ, then we continue along the segment γ until we reach the next node, where we follow the procedure described for the first node. We point out that the slope of the segment γ is necessarily strictly smaller than the slope of the straight line followed before hitting γ.

In this way a graph of $h(x)$ with the required properties is constructed. Observe that if the graph of $h(x)$ has infinitely many points in common with Γ, then clearly $h(x) \to \infty$ as $x \to \infty$. But if there are only finitely many points

in common with Γ, then $h(x)$ is eventually linear with a positive slope, hence $h(x) \to \infty$ as $x \to \infty$.

$\square$

Proof of Theorem 3.6. Defining $e_n = 1/\tau_n$, there exists by Lemma 3.7 a concave, strictly increasing sequence (d_n) with $d_0 = 1$ and $\lim d_n = \infty$ and such that $d_n \leq e_n$. Moreover, we may assume that $d_n \leq n+1$ by replacing d_n by $\min(d_n, n+1)$. In this way we may also assume that $d_2 \leq 3$. This implies that there exists a decreasing sequence of positive numbers $c_k, k \geq 1$ such that $c_1 \leq 1$ and

$$d_{2n} = 1 + 2\sum_{k=1}^{n} c_k.$$

In fact, we define

$$c_1 = (d_2 - 1)/2, \quad c_n = (d_{2n} - d_{2n-2})/2, \quad n \geq 2,$$

so (c_n) is decreasing because d_{2n} is concave.

Let the sequence η_k be defined by

$$1 - \eta_1 = \sqrt{c_1}, \quad 1 - \eta_k = \sqrt{\frac{c_k}{c_{k-1}}}, \quad k \geq 2. \tag{31}$$

Then $\eta_k \geq 0$ and

$$d_{2n} = 1 + 2\sum_{k=1}^{n}\prod_{l=1}^{k}(1 - \eta_l)^2. \tag{32}$$

Define the sequence ξ_k by

$$\xi_{2k-1} = \xi_{2k} = \eta_k, \quad k \geq 1.$$

Inspired by formula (30) we finally define a positive sequence (b_n) by $b_0 = 1$ and

$$b_n = \prod_{k=1}^{n}(1 - \xi_k)^{-1}\left[2 + \sum_{k=1}^{n-1}\prod_{l=1}^{k}(1 - \xi_l)\right], \quad n \geq 1.$$

Then we get for $n \geq 1$

$$\begin{aligned} b_{2n} &\leq \prod_{k=1}^{n}(1 - \eta_k)^{-2}\left[3 + 2\sum_{k=1}^{n-1}\prod_{l=1}^{k}(1 - \eta_l)^2\right] \\ &= 2\frac{2 + d_{2n-2}}{d_{2n} - d_{2n-2}} < 2\frac{2 + d_{2n}}{d_{2n} - d_{2n-2}}, \end{aligned}$$

where we used formula (32). This gives

$$\frac{1}{b_{2n}} > \frac{d_{2n} - d_{2n-2}}{2(2 + d_{2n})},$$

and since d_{2n} tends to infinity we get

$$\sum_{n=1}^{\infty} \frac{1}{b_{2n}} = \infty. \tag{33}$$

In fact, assuming the contrary we get

$$\infty > \sum_{n=1}^{\infty} \frac{1}{b_{2n}} > \sum_{n=1}^{\infty} \frac{d_{2n} - d_{2n-2}}{2(2 + d_{2n})},$$

so there exists $N \in \mathbb{N}$ such that for all $p \in \mathbb{N}$

$$\frac{1}{2} \geq \sum_{n=N+1}^{N+p} \frac{d_{2n} - d_{2n-2}}{2 + d_{2n}} > \sum_{n=N+1}^{N+p} \frac{d_{2n} - d_{2n-2}}{2 + d_{2N+2p}} = \frac{d_{2N+2p} - d_{2N}}{2 + d_{2N+2p}},$$

but the right-hand side converges to 1 for $p \to \infty$, which is a contradiction.

The positive sequence (b_n) defines a system of orthonormal polynomials via (22). The corresponding symmetric probability measure is determinate by Carleman's theorem because of (33). Moreover, by Proposition 3.4 and formula (32) we get

$$\lambda_{2N} \geq \lambda_{2N+1} \geq \frac{1}{2d_{2N}} \geq \frac{1}{2e_{2N}} = \frac{1}{2}\tau_{2N} \geq \frac{1}{2}\tau_{2N+1}. \quad \square$$

4 The indeterminate case

Let (s_n) be the moment sequence (1). The inequality

$$\sum_{n,m=0}^{N} s_{n+m} a_n \overline{a_m} \geq c \sum_{k=0}^{N} |a_k|^2, \quad a \in \mathbb{C}^{N+1}$$

can be rewritten

$$\int \left| \sum_{k=0}^{N} a_k x^k \right|^2 d\mu(x) \geq c \sum_{k=0}^{N} |a_k|^2. \tag{34}$$

If we write

$$\sum_{k=0}^{N} a_k x^k = \sum_{n=0}^{N} c_n P_n(x)$$

and use (8), then (34) takes the form

$$\sum_{n=0}^{N} |c_n|^2 \geq c \sum_{k=0}^{N} \left| \sum_{n=k}^{N} b_{k,n} c_n \right|^2 .$$

This immediately gives the following result:

Lemma 4.1. *The eigenvalues λ_N are bounded below by a constant $c > 0$ if and only if the upper triangular matrix $\mathcal{B} = (b_{k,n})$ given by* (9) *corresponds to a bounded operator on ℓ^2 of norm $\leq 1/\sqrt{c}$.*

Recalling that the indeterminate case was characterized in [4] by λ_N being bounded below by a positive constant, we see that the indeterminate case is characterized by the boundedness of the operator $\mathcal{B}$. For a characterization of the lower boundedness of λ_N in a more general setting see [7]. As noticed in [4, Remark, p. 72], the indeterminacy is also equivalent to the boundedness of the matrix $\mathcal{K}$, cf. (11), which is automatically in trace class if it is bounded.

Concerning the matrices $\mathcal{A}, \mathcal{K}, \mathcal{B}$, given by (14), (11),(9) respectively, we have:

Proposition 4.2. *Assume that μ is indeterminate. Then the following matrix equations hold*

(i) $\mathcal{K} = \mathcal{B}^* \mathcal{B}$,

(ii) $\mathcal{A} = \mathcal{B}\mathcal{B}^*$.

$\mathcal{A}, \mathcal{B}, \mathcal{K}$ are of trace class and

$$tr(\mathcal{A}) = tr(\mathcal{K}) = \rho_0,$$

where ρ_0 is defined in (15).

Furthermore, the sequence

$$c_k = \sqrt{a_{k,k}} = \left(\sum_{n=k}^{\infty} |b_{k,n}|^2 \right)^{1/2}, \tag{35}$$

satisfies

$$\lim_{k\to\infty} k \sqrt[k]{c_k} = 0, \tag{36}$$

and the matrix $\mathcal{A} = (a_{j,k})$ has the following property

$$\sum_{j,k=0}^{\infty} |a_{j,k}|^{\varepsilon} < \infty \tag{37}$$

for any $\varepsilon > 0$.

Proof. From (8) we have

$$b_{k,n} = \frac{1}{2\pi i}\int_{|z|=r} P_n(z)z^{-(k+1)}\,dz = r^{-k}\frac{1}{2\pi}\int_0^{2\pi} P_n(re^{it})e^{-ikt}\,dt. \tag{38}$$

Consider $r = 1$. Then, by Parseval's identity we have

$$\sum_{n=0}^{N}\sum_{k=0}^{n}|b_{k,n}|^2 = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{N}|P_n(e^{it})|^2\,dt. \tag{39}$$

Therefore, in the indeterminate case the matrix $\mathcal{B}$ is Hilbert-Schmidt with Hilbert-Schmidt norm $\rho_0^{1/2}$, cf. (15). Hence both $\mathcal{B}^*\mathcal{B}$ and $\mathcal{B}\mathcal{B}^*$ are of trace class with trace ρ_0. Formula (i) of Proposition 4.2 is an immediate consequence of Parseval's identity.

We know that $K_N(z,w)$ defined in (10) converges to $K(z,w)$, locally uniformly in $\mathbb{C}^2$, hence

$$a_{j,k}^{(N)} = \sum_{n=\max(j,k)}^{N} b_{j,n}b_{k,n} \to a_{j,k} \tag{40}$$

for each pair (j,k). The series

$$\sum_{n=\max(j,k)}^{\infty} b_{j,n}b_{k,n} = \sum_{n=0}^{\infty} b_{j,n}b_{k,n}$$

is absolutely convergent for each pair (j,k) because $\mathcal{B}$ is Hilbert-Schmidt, so (40) implies (ii).

Defining

$$c_k = ||\mathcal{B}^*\delta_k|| = \left(\sum_{n=k}^{\infty}|b_{k,n}|^2\right)^{1/2}, \tag{41}$$

where $\delta_k, k = 0,1,\ldots$ denotes the standard orthonormal basis in ℓ^2, we have the following estimate for $r > 1$ using the Cauchy-Schwarz inequality

$$\left(\sum_{k=0}^{\infty}c_k\right)^2 \le \sum_{k=0}^{\infty}r^{-2k}\sum_{k=0}^{\infty}r^{2k}c_k^2 = \frac{r^2}{r^2-1}\sum_{k=0}^{\infty}r^{2k}\sum_{n=k}^{\infty}|b_{k,n}|^2.$$

However, by (38) and by Parseval's identity we have

$$\sum_{k=0}^{\infty}r^{2k}\sum_{n=k}^{\infty}|b_{k,n}|^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}r^{2k}|b_{k,n}|^2 = \sum_{n=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}|P_n(re^{it})|^2\,dt. \tag{42}$$

Let now

$$P(z) = \left(\sum_{n=0}^{\infty}|P_n(z)|^2\right)^{1/2}, \quad z \in \mathbb{C}. \tag{43}$$

We finally get

$$\sum_{k=0}^{\infty} c_k \leq \frac{r}{\sqrt{r^2-1}} \left(\frac{1}{2\pi} \int_0^{2\pi} |P(re^{it})|^2 \, dt \right)^{1/2} < \infty,$$

but since

$$\langle |\mathcal{B}^*|\delta_k, \delta_k \rangle \leq || \, |\mathcal{B}^*|\delta_k \, || = ||\mathcal{B}^* \delta_k||$$

this shows that $|\mathcal{B}^*|$ and hence $\mathcal{B}$ is of trace class.

For a given $\varepsilon > 0$ we have $P(z) \leq C_\varepsilon e^{\varepsilon|z|}$ by a theorem of M. Riesz, cf. [1, Th. 2.4.3], hence by (41) and (42)

$$\sum_{k=0}^{\infty} r^{2k} c_k^2 \leq C_\varepsilon^2 e^{2\varepsilon r}.$$

For $r = k/\varepsilon$ we get in particular

$$\left(\frac{k}{\varepsilon} \right)^{2k} c_k^2 \leq C_\varepsilon^2 e^{2k},$$

hence

$$\limsup_{k \to \infty} k \sqrt[k]{c_k} \leq e\varepsilon,$$

which shows (36).

Using $|a_{j,k}| \leq c_j c_k$, it is enough to prove that $\sum_{k=0}^{\infty} c_k^\varepsilon < \infty$ for $0 < \varepsilon < 1$, which is weaker than (36). □

For a sequence $\alpha = (\alpha_n) \in \ell^2$ we consider the function

$$F_\alpha(z) = \sum_{n=0}^{\infty} \alpha_n P_n(z) = \sum_{n=0}^{\infty} \beta_n z^n, \tag{44}$$

which is an entire function of minimal exponential type because

$$|F_\alpha(z)| \leq ||\alpha|| P(z),$$

where $P(z)$ is given by (43). The following result is a straightforward consequence of (44).

Proposition 4.3. *The sequence of coefficients $\beta = (\beta_n)$ of the power series of F_α belongs to ℓ^2 and is given by $\beta = \mathcal{B}\alpha$. The operator $\mathcal{B} : \ell^2 \to \ell^2$ is one-to-one with dense range $\mathcal{B}(\ell^2)$.*

For a compact operator T on ℓ^2 we denote by $\sigma_n(T), n = 0, 1, \ldots$ the singular values of T in decreasing order, i.e.

$$\sigma_n(T) = \min_{V \subset \ell^2, \dim V = n} \quad \max_{\|v\|=1, \ v \perp V} \|Tv\|. \tag{45}$$

Theorem 4.4. *Assume that μ is indeterminate. Let*

$$\lambda_N = \lambda_{N,0} \le \lambda_{N,1} \le \ldots \le \lambda_{N,N}$$

denote the $N+1$ eigenvalues of $\mathcal{H}_N$ and let

$$\lambda_{\infty,n} = \lim_{N\to\infty} \lambda_{N,n}.$$

For $0 \le n \le N$ we have

$$\sigma_n(\mathcal{A}) = \sigma_n(\mathcal{B}^*)^2 \ge \frac{1}{\lambda_{\infty,n}} \ge \frac{1}{\lambda_{N,n}} \tag{46}$$

and

$$\lim_{n\to\infty} n^2 \sqrt[n]{\sigma_n(\mathcal{A})} = 0, \quad \lim_{n\to\infty} \frac{\sqrt[n]{\lambda_{\infty,n}}}{n^2} = \infty. \tag{47}$$

Proof. By (45) we get

$$\sigma_n(\mathcal{B}^*) \le \max_{\|v\|=1,\, v\perp\delta_0,\ldots,\delta_{n-1}} \|\mathcal{B}^* v\|.$$

Let Π_n denote the projection onto $\{\delta_0, \ldots, \delta_{n-1}\}^\perp$. Thus by (41)

$$\sigma_n(\mathcal{B}^*) \le \|\mathcal{B}^*\Pi_n\| \le \left(\sum_{k=n}^{\infty} c_k^2\right)^{1/2}.$$

On the other hand, for $r \ge 1$ we have

$$\sum_{k=n}^{\infty} c_k^2 \le \sum_{k=n}^{\infty} c_k^2 \frac{(k!)^2 r^k}{(n!)^2 r^n} \le \frac{S(r)}{(n!)^2 r^n},$$

where

$$S(r) := \sum_{k=0}^{\infty} (k! c_k)^2 r^k < \infty$$

because of (36) and $\sqrt[k]{k!} \sim k/e$, which holds by Stirling's formula. Therefore

$$\sigma_n(\mathcal{B}^*)^2 \le \frac{S(r)}{(n!)^2 r^n},$$

and since $\sigma_n(\mathcal{B}^*) = \sqrt{\sigma_n(\mathcal{B}\mathcal{B}^*)}$ we get

$$\sigma_n(\mathcal{A}) = \sigma_n(\mathcal{B}\mathcal{B}^*) \le \frac{S(r)}{(n!)^2 r^n}, \quad r \ge 1, \tag{48}$$

which proves the first assertion of (47).

Let Pr_N denote the projection in ℓ^2 onto span$\{\delta_0, \ldots, \delta_N\}$. We then have

$$(\mathcal{B}Pr_N)(\mathcal{B}Pr_N)^* = \mathcal{B}Pr_N\mathcal{B}^* \leq \mathcal{B}\mathcal{B}^*,$$

and therefore for $n \leq N$

$$\sigma_n(\mathcal{B}\mathcal{B}^*) \geq \sigma_n((\mathcal{B}Pr_N)(\mathcal{B}Pr_N)^*) = \sigma_n(\mathcal{B}_N\mathcal{B}_N^*) = \sigma_n(\mathcal{H}_N^{-1}),$$

where the last equality follows by Theorem 1.1. The matrix $\mathcal{H}_N^{-1}$ is positive definite, so its singular values are the eigenvalues which are the reciprocals of the eigenvalues of $\mathcal{H}_N$, i.e. $\sigma_n(\mathcal{H}_N^{-1}) = 1/\lambda_{N,n}$. This gives (46) and the second assertion in (47) follows. □

Theorem 4.5. *The trace class operator $\mathcal{A} : \ell^2 \to \ell^2$ is positive with spectrum*

$$\sigma(\mathcal{A}) = \{0\} \cup \{\lambda_{\infty,n}^{-1} \mid n = 0, 1, \ldots\}.$$

Proof. We will consider $\mathcal{A}^{(N)} = (a_{j,k}^{(N)})$ and $\mathcal{B}_N$ as finite rank operators on ℓ^2 by adding zero rows and columns. Clearly, $\mathcal{B}_N$ tends to $\mathcal{B}$ in the Hilbert-Schmidt norm, and therefore $\mathcal{A}^{(N)} = \mathcal{B}_N\mathcal{B}_N^*$ tends to $\mathcal{A} = \mathcal{B}\mathcal{B}^*$ in the trace norm.

The result now follows since the spectrum of $\mathcal{A}^{(N)}$ consists of the numbers $\lambda_{N,n}^{-1}, n = 0, 1, \ldots, N$, by Theorem 1.1. □

5 The Stieltjes-Wigert polynomials

For $0 < q < 1$ we consider the moment sequence $s_n = q^{-(n+1)^2/2}$ given by

$$\frac{1}{\sqrt{2\pi \log(1/q)}} \int_0^\infty x^n \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right) dx. \tag{49}$$

We call it the Stieltjes-Wigert moment sequence because Stieltjes proved that it is indeterminate (he considered the special value $q = \frac{1}{2}$) and Wigert [23] found the corresponding orthonormal polynomials

$$P_n(x; q) = (-1)^n \frac{q^{\frac{n}{2}+\frac{1}{4}}}{\sqrt{(q;q)_n}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2+\frac{k}{2}} x^k. \tag{50}$$

Here we have used the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

involving the q-shifted factorial

$$(z;q)_n = \prod_{k=1}^{n} (1 - zq^{k-1}), \quad z \in \mathbb{C}, n = 0, 1, \ldots, \infty.$$

We refer to [13] for information about this notation and q-series. We have followed the normalization used in Szegő [19], where $s_0 = 1/\sqrt{q}$. The Stieltjes-Wigert moment problem has been extensively studied in [11] using a slightly different normalization.

Lemma 5.1. *The double sum*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{j,n}b_{k,n}s_{k+l}$$

is absolutely convergent for each $j, l \geq 0$ and

$$|a_{j,k}| \leq \frac{q^{j^2+k^2}}{(q;q)_j(q;q)_k(q;q)_\infty^2}.$$

Moreover, $\mathcal{A}\mathcal{H} = \mathcal{H}\mathcal{A} = I$.

Proof. We find

$$|b_{j,n}b_{k,n}| = \frac{(q;q)_n}{(q;q)_j(q;q)_k(q;q)_{n-j}(q;q)_{n-k}} q^{n+j^2+k^2+\frac{j+k+1}{2}},$$

hence for $j \geq k$

$$\begin{aligned}
|a_{j,k}| &\leq \frac{q^{j^2+k^2+\frac{j+k+1}{2}}}{(q;q)_j(q;q)_k} \sum_{n=j}^{\infty} \frac{(q;q)_n}{(q;q)_{n-j}(q;q)_{n-k}} q^n \\
&= \frac{q^{j^2+k^2+\frac{j+k+1}{2}}}{(q;q)_j(q;q)_k} \sum_{p=0}^{\infty} \frac{(q;q)_{j+p}}{(q;q)_p(q;q)_{j-k+p}} q^{j+p} \\
&= \frac{q^{j^2+k^2+\frac{j+k+1}{2}+j}}{(q;q)_{j-k}(q;q)_k} \sum_{p=0}^{\infty} \frac{(q^{j+1};q)_p}{(q;q)_p(q^{j-k+1};q)_p} q^p \\
&\leq \frac{q^{j^2+k^2}}{(q;q)_j(q;q)_k} \sum_{p=0}^{\infty} \frac{q^p}{(q;q)_p(q;q)_\infty} = \frac{q^{j^2+k^2}}{(q;q)_j(q;q)_k(q;q)_\infty^2},
\end{aligned}$$

where we have used the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1 \tag{51}$$

with $a = 0, z = q$. By symmetry the estimate holds for all pairs j, k. Since $s_{k+l} = q^{-(k+l+1)^2/2}$ it is clear that the double sum is absolutely convergent.

By Lemma 1.2 we then have

$$I = \mathcal{B}(\mathcal{B}^*\mathcal{H}) = (\mathcal{B}\mathcal{B}^*)\mathcal{H} = \mathcal{A}\mathcal{H},$$

and we clearly have $\mathcal{H}\mathcal{A} = \mathcal{A}\mathcal{H}$. □

From (50) we get

$$P_n(0;q) = (-1)^n \frac{q^{\frac{n}{2}+\frac{1}{4}}}{\sqrt{(q;q)_n}}, \tag{52}$$

hence by (51)

$$\sum_{n=0}^{\infty} P_n^2(0;q) = \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{(q;q)_n} = \frac{\sqrt{q}}{(q;q)_\infty}. \tag{53}$$

The matrix $\mathcal{K} = (\kappa_{j,k})$ defined in (11) is given by

$$\kappa_{j,k} = (-\sqrt{q})^{j+k} \frac{\sqrt{q}}{\sqrt{(q;q)_j (q;q)_k}} \sum_{p=0}^{\min(j,k)} \begin{bmatrix} j \\ p \end{bmatrix}_q \begin{bmatrix} k \\ p \end{bmatrix}_q q^{2p^2+p}, \tag{54}$$

hence

$$\rho_0 = \sum_{k=0}^{\infty} \kappa_{k,k} = \sqrt{q} \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \sum_{p=0}^{k} \begin{bmatrix} k \\ p \end{bmatrix}_q^2 q^{2p^2+p}, \tag{55}$$

in accordance with [4], which also contains other expressions for ρ_0. From (53),(18) with $z_0 = 0$ and [4, Theorem 1.2] we get

$$1/\rho_0 \le \lim_{N\to\infty} \lambda_N < \frac{(q;q)_\infty}{\sqrt{q}}.$$

From the general theory we know that the Stieltjes-Wigert moment sequence has an N-extremal solution ν_0, which has the mass $c = (q;q)_\infty/\sqrt{q}$ (=the reciprocal of the value in (53)) at 0. It is a discrete measure concentrated at the zeros of the entire function

$$D(z) = z \sum_{n=0}^{\infty} P_n(0;q) P_n(z;q).$$

It is also known by a result of Stieltjes, that the measure $\tilde{\mu} = \nu_0 - c\varepsilon_0$ is determinate, cf. e.g. [5, Theorem 7]. The moment sequence $(\tilde{s}_n)$ of $\tilde{\mu}$ equals the Stieltjes-Wigert moment sequence except for the zeroth moment, i.e.

$$\tilde{s}_n = \begin{cases} (1-(q;q)_\infty)/\sqrt{q} & \text{if } n = 0 \\ q^{-(n+1)^2/2} & \text{if } n \ge 1, \end{cases}$$

and similarly the corresponding Hankel matrices $\mathcal{H}$ and $\tilde{\mathcal{H}}$ differ only at the entry $(0,0)$.

We shall prove

Theorem 5.2. *The smallest eigenvalue $\tilde{\lambda}_N$ corresponding to the measure $\tilde{\mu}$ tends exponentially to zero in the sense that there exists a constant $A > 0$ such that*

$$\tilde{\lambda}_N \le A q^N.$$

The proof of Theorem 5.2 depends on the quite remarkable fact that it is possible to find an explicit formula for the corresponding orthonormal polynomials which will be denoted $\tilde{P}_n(x;q)$. It is a classical fact, cf. [1, p.3], that the orthonormal polynomials (P_n) corresponding to a moment sequence (s_n) are given by the formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \det \begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}, \tag{56}$$

where $D_n = \det(\mathcal{H}_n)$. In this way Wigert calculated the polynomials $P_n(x;q)$, and we shall follow the same procedure for $\tilde{P}_n(x;q)$. Writing

$$\tilde{P}_n(x;q) = \sum_{k=0}^{n} \tilde{b}_{k,n} x^k, \tag{57}$$

we have

Theorem 5.3. *For* $0 \le k \le n$

$$\tilde{b}_{k,n} = \tilde{C}_n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2+\frac{k}{2}} \left[1-(1-q^k)(q^{n+1};q)_\infty\right], \tag{58}$$

where

$$\tilde{C}_n = \frac{(-1)^n q^{\frac{n}{2}+\frac{1}{4}}}{\sqrt{(q;q)_n}\sqrt{(1-(q^n;q)_\infty)(1-(q^{n+1};q)_\infty)}}, \tag{59}$$

i.e.

$$\tilde{b}_{k,n} = b_{k,n} \frac{1-(1-q^k)(q^{n+1};q)_\infty}{\sqrt{(1-(q^n;q)_\infty)(1-(q^{n+1};q)_\infty)}}, \tag{60}$$

where $b_{k,n}$ *denote the coefficients of* $P_n(x;q)$. *Moreover,*

$$\tilde{D}_n = D_n(1-(q^{n+1};q)_\infty), \tag{61}$$

where $D_n = \det \mathcal{H}_n$, $\tilde{D}_n = \det \tilde{\mathcal{H}}_n$.

Proof. We first recall the Vandermonde determinant

$$V_n(x_1,\dots,x_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i). \tag{62}$$

For an $n \times n$-matrix $(a_{j,k}),\ j,k = 1 \dots n$ with non-zero elements in the first row and column we have

$$\det(a_{j,k}) = \left(\prod_{j=1}^{n} a_{j,1}\right)\left(\prod_{k=1}^{n} a_{1,k}\right)\det\left(\frac{a_{j,k}}{a_{j,1}a_{1,k}}\right),$$

and if $a_{j,k} = q^{-(j+k-1)^2/2}$, $j = k = 1,\dots,n+1$, where $0 < q < 1$, we get in particular

$$D_n = \det(q^{-(j+k-1)^2/2}) = \left(\prod_{j=1}^{n+1} q^{-j^2/2}\right)^2 \det(q^{-(j-1)(k-1)+1/2}), \tag{63}$$

hence using $S_n = \sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$

$$D_n = q^{-S_{n+1}+(n+1)/2} V_{n+1}(1, q^{-1}, \dots, q^{-n}). \tag{64}$$

By (62) we get

$$V_{n+1}(1, q^{-1}, \dots, q^{-n}) = \prod_{i=0}^{n}\prod_{j=i+1}^{n}\frac{1}{q^j}(1-q^{j-i}) = \prod_{i=0}^{n} q^{-(n-i)(n+i+1)/2}(q;q)_{n-i},$$

and after some reduction

$$V_{n+1}(1, q^{-1}, \dots, q^{-n}) = q^{-S_n}\prod_{j=1}^{n}(q;q)_j. \tag{65}$$

We denote by $A_{r+1,p+1}$ respectively $\tilde{A}_{r+1,p+1}$ the cofactor of the entry $(r+1,p+1)$ of the Hankel matrix $\mathcal{H}_n = (q^{-(j+k-1)^2/2})$ respectively $\tilde{\mathcal{H}}_n$, where $r,p = 0,1,\dots,n$. When $r = 0$ or $p = 0$ we clearly have $A_{r+1,p+1} = \tilde{A}_{r+1,p+1}$. For $0 < p < n$ we get

$$A_{n+1,p+1} = (-1)^{n-p}\det\left(q^{-(j+k-1)^2/2} \mid \begin{smallmatrix} j=1,\dots,n \\ k=1,\dots,n+1;\,k\neq p+1\end{smallmatrix}\right)$$

$$= (-1)^{n-p}\prod_{j=1}^{n} q^{-j^2/2}\prod_{\substack{k=1\\k\neq p+1}}^{n+1} q^{-k^2/2}\det\left(q^{-(j-1)(k-1)+1/2} \mid \begin{smallmatrix} j=1,\dots,n \\ k=1,\dots,n+1;\,k\neq p+1\end{smallmatrix}\right)$$

$$= (-1)^{n-p} q^{-S_{n+1}+((n+1)^2+(p+1)^2+n)/2} V_n(1, q^{-1}, \dots, q^{-(p-1)}, q^{-(p+1)}, \dots, q^{-n}).$$

However,

$$\begin{aligned} &V_{n+1}(1, q^{-1}, \dots, q^{-n}) \\ &= V_n(1, q^{-1}, \dots, q^{-(p-1)}, q^{-(p+1)}, \dots, q^{-n})\prod_{j=0}^{p-1}(q^{-p}-q^{-j})\prod_{j=p+1}^{n}(q^{-j}-q^{-p}) \\ &= V_n(1, q^{-1}, \dots, q^{-(p-1)}, q^{-(p+1)}, \dots, q^{-n})(q;q)_p(q;q)_{n-p}q^{-(n^2+p^2+n-p)/2}, \end{aligned}$$

so we finally get

$$A_{n+1,p+1}/D_n = (-1)^n q^{(n+1)(n+1/2)} \frac{(-1)^p q^{p(p+1/2)}}{(q;q)_p (q;q)_{n-p}}, \quad 0 < p < n. \tag{66}$$

It can be verified that this formula also holds for $p = 0$ and $p = n$.

Using (56) it is now easy to verify formula (50) for the Stieltjes-Wigert polynomials $P_n(x;q)$.

Expanding after the first column we get

$$\tilde{D}_n = D_n - cA_{1,1}, \quad c = (q;q)_\infty / \sqrt{q},$$

and a calculation as above leads to

$$A_{1,1} = q^{-S_{n+2}+5+9(n/2)} V_n(1, q^{-1}, \dots, q^{-(n-1)}),$$

which gives (61). Moreover, for $0 < p \leq n$ we find

$$\tilde{A}_{n+1,p+1} = A_{n+1,p+1} - c(-1)^{n-p} \det \left(q^{-(j+k+1)^2} \mid {}^{\,j=1,\dots,n-1}_{\,k=1,\dots,n;\, k\neq p} \right),$$

and the last determinant can be calculated to be

$$\frac{D_{n-1}}{(q;q)_{n-p}(q;q)_{p-1}} q^{-n^2-(n-1)/2+p(p+1/2)}.$$

This leads to

$$\frac{\tilde{A}_{n+1,p+1}}{\tilde{D}_n} = \frac{A_{n+1,p+1}}{D_n} \frac{1-(1-q^p)(q^{n+1};q)_\infty}{1-(q^{n+1};q)_\infty}. \tag{67}$$

It can be verified that this formula also holds for $p = 0$ because of (61), and it is now easy to establish (60). $\square$

Proof of Theorem 5.2 By Lemma 2.1 we get

$$\tilde{\lambda}_N \leq (\tilde{P}_N(0;q))^{-2} = \frac{(q;q)_N (1-(q^{N+1};q)_\infty)(1-(q^N;q)_\infty)}{q^{N+1/2}}.$$

From the power series expansion of the entire function $(z;q)_\infty$ we have

$$1-(z;q)_\infty \sim \frac{z}{1-q}, \quad z \to 0,$$

hence

$$1-(q^N;q)_\infty \sim \frac{q^N}{1-q}, \quad N \to \infty, \tag{68}$$

and therefore

$$(\tilde{P}_N(0;q))^{-2} \sim \frac{(q;q)_\infty}{(1-q)^2} q^{N+1/2}, \quad N \to \infty,$$

which proves the statement of the theorem.

Remark 5.4. The measure $\tilde{\mu}$ is determinate of index 0, cf. [6], so by Corollary 2.1 in [7] we know that the next smallest eigenvalue $\tilde{\lambda}_{N,1}$ of $\tilde{\mathcal{H}}_N$ is bounded below.

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